Harmonic maps into periodic flag manifolds and into loop groups

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Abstract. *In this note we discuss harmonic maps into periodic flag manifolds and into Loop groups. We also discuss the stability of some maps called Eells-Wood-Uhlenbeck into such manifolds.*

§ 1. INTRODUCTION

Due to the enormous difficulties involving Gauge theories in 4 dimensions, considerable attention has been given to 2-dimensional sigma models (harmonic maps!) which is hoped to share some of the important qualitative properties of the 4 dimensional theory.

It is well-known that both functionals have several common features. For example: they are conformally invariant when the dimension n of the domain manifold is 2 in the case of the energy functional and $n = 4$ for the Yang-Mills functional. They also have similar compaticity theorems like Sacks-Uhlenbeck [26] and Uhlenbeck's results contained in [28] and [29].

They also have special solutions (instantons) which are ± - holomorphic maps (when the metrics are Kähler) for the energy functional and (anti)–self–dual connections for the Yang-Mills funcitonal. Furhtermore Atiyah [3] and Donaldson [12] have shown that the instantons in both theories can be naturally identified. The rough idea of this identification is: Ward and Penrose have shown

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that instantons on S^4 are in natural $1-1$ correspondence with holomorphic bundles over $\mathbb{C}P^3$. But Donaldson in [12] has shown that such holomorphic bundles over $\mathbb{C}P^3$ are in natural $1-1$ correspondence with holomorphic bundles over $\mathbb{C}P^2$ plus some extra technical condition. On the other hand, Atiyah in [3] has shown that holomorphic bundles over $\mathbb{C}P^2$ plus the extra condition are in $1-1$ correspondence with holomorphic bundles over $\mathbb{C}P^1 \times \mathbb{C}P^1$ plus a similar extra condition found by Donaldson. But holomorphic bundles over $\mathbb{C}P^1 \times \mathbb{C}P^1$ plus the extra technical condition are in $1-1$ correspondence with holomorphic maps $f : S^2 \to \Omega(G)$. We can summarize this discussion into the following diagram:

instantons on S^4 Ward $\leftrightarrow$ Penrose holomorphic bundles over $\mathbb{C}P^3$ (if we consider the Penrose fibration $S2 \ldots \to \mathbb{C}P^3 \to S^4$)

$\updownarrow$ Donaldson holomorphic bundles over $\mathbb{C}P^2 + \ldots$

$\updownarrow$ Atiyah holomorphic bundles over $\mathbb{C}P^1 \times \mathbb{C}P^1 + \ldots$

holomorphic maps $f : S^2 \to \Omega(G) \leftrightarrow$

A basic variational problem for the Yang-Mills functional is the classification of all Yang-Mills connections on a compact 4-dimensional manifold in terms of instantons solutions. Quite recently L. Sibner, R. Sibner and K. Uhlenbeck [27] have announced the existence of an infinite number of Yang-Mills connections over an $SU(2)$-bundle on S^4 that are not a local minimum for the Yang-Mills functional, so they can not be instantons according to [5]. This theorem may suggest a very strong difference between the variational aspects of the energy and the Yang-Mills functionals.

Examples of Yang-Mills connections on $S^2 \times S^2$ are known are not instantons. See [30] for more details. Before [27] all the non-instantons solutions were obtained in a more or less standard way via algebraic and geometric methods.

We can translate this variational problem to the energy functional level in the following way: try to classify harmonic maps $\phi : M^2 \to (\Omega(U(n))$, Kähler metric) in terms of holomorphic maps $\phi : M^2 \to \Omega(U(n))$, where M^2 is a compact Riemann surface. For example it is a difficult and open problem to find $\phi : S^2 \to (\Omega(SU(2))$, Kähler metric) harmonic but not holomorphic. We will in this note prove some results that seem to indicate that both critical point theories are related.

In section 2 we discuss the almost complex structures on $\Omega(G)$ and the natural Kähler metric in the loop space and into the periodic flag manifold.

In section 3 we compute the Euler-Lagrange equations for harmonic maps from compact Riemann surfaces into periodic flag manifolds by using the geometric approach suggested by Atiyah in [2], and derive some related results.

In section 4 we define some basic harmonic maps into periodic flag manifolds which are a generalization of the ones found by Eells and Wood in [13]. We discuss the stability of such maps with respect to a large class of invariant metrics on $F\ell^{(n)}$. We prove the existence of «saddles» in the space.

In section 5 we study harmonic maps from $T^2 = S^1 \times S^1$ into $F\ell^{(n)}$ which are equivariant with respect to a circle action. We can find families of harmonic maps that are not holomorphic with respect to any almost complex structure on $\Omega(G)$.

Finally, in the last section we discuss the connection of the former results wih the study of harmonic maps into Loop groups as in [22].

The previous results seem to indicate a close relationship among the set of invariant metrics on $\Omega(U(n))$ and the moduli space of $U(n)$-connections on S^4.

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§2. SOME REMARKS CONCERNING TO THE GEOMETRY OF $\Omega(G)$

We define the free group of loops $L(U(n))$ as the set of smooth maps from S^1 to $U(n)$, and $\Omega(U(n))$ as the subgroup of $L(U(n))$ formed of maps f such that $f(1) = I$. We will be talking in this paragraph about $\Omega(U(n))$.

The simplest case is when $G = U(1)$. Then $\Omega(U(1))$ has components indexed according to the winding number and each component can be identified with the space of functions $f : S^1 \to \mathbb{R}$ such that $f(1) = 0$. The Fourier series of such a function is $\phi = \sum_{n=-\infty}^{\infty} a_n Z^n$, $a_{-n} = \bar{a}_n$, $\sum_n a_n = 0$, therefore the coefficients a_n, for $n \geqslant 0$ determine ϕ completely. Hence, each component of $\Omega(U(1))$ becomes a complex vector space of infinite dimension.

For non-abelian G, $\Omega(G)$ is not a vector space anymore. However, is still is an infinite dimensional manifold and we can use Fourier series to introduce complex coordinates. We known that $\Omega(G)_I$ is equal to $\Omega(g)$ and can be represented in Fourier series as the set of functions $\phi = \sum_{n=-\infty}^{\infty} a_n Z^n$, $a_{-n} = -a_n^*$, $\sum a_n = 0$ where $a_n \in G_{\mathbb{C}}$. If $G = U(m)$ then $a_n \in \mathbb{C}_m$ and a_n^* is the transpose conjugate matrix. Furthermore, since $\Omega(U(m))$ is equal to

$L(U(m))/U(m)$, if we denote by p the space $\Omega(U(m))_I$, then $L(u(m)) = p \oplus u(m)$ so if $\phi \in \Omega(U(m))_I = \Omega(u(m))$, $\phi = \sum_{n=-\infty}^{\infty} a_n Z^n$, $a_n \in u(m)_{\mathbb{C}}$ and $a_0 = 0$. So $\Omega(U(m))_I$ becomes an infinite dimensional complex vector space.

Now, we can define several almost complex structures on $\Omega(G)$, namely if $\phi \in \Omega(G)_I$ we define $J(\phi) = \sum_{n=-\infty}^{\infty} \alpha_n \sqrt{-1}\, a_n Z^n$, where $\alpha_n = \pm 1$ and $\alpha_{\ n} = \mp 1$. The almost complex structure obtained by making $\alpha_n = 1 \ \forall\ n > 0$ is integrable and is called the canonical almost complex structure. Notice that the almost complex structure obtained by making $\alpha_n = (-1)^n$, $n > 0$ is not integrable, and there are some interesting results concerning this almost complex structure (See [4] for more details). If we fix one almost complex structure J, we can translate it for all tangent spaces hence $(\Omega(G), J)$ becomes an almost complex manifold of infinite dimension.

We also have an alternative and useful description of the canonical Kähler structure on $\Omega(G)$. To define a Hermitian metric on $\Omega(G)$ is again enough to define it in $\Omega(G)_I$ and translate it via the group action. In section 3 we will exhibit several natural invariant metrics on $\Omega(G)$, but the most natural it seems to be given with respect to the Fourier coefficients by $\sum_{n>0} n\,tr(a_n a_n^*)$ where a_n is regarded as a matrix. A reason for this metric being natural and important relies on the fact that it is Kähler. The symplectic form associated is given by:

$$(\phi, \psi) = \frac{1}{2\Pi} \int_0^{2\Pi} \langle \phi'(\theta), \psi(\theta) \rangle\, do, \text{ where } \langle\ ,\ \rangle$$

is given by the Killing form.

§3. HARMONIC MAPS INTO PERIODIC FLAG MANIFOLDS

In this paragraph we will consider throughout the Hilbert space $H^{(n)} = L^2(S^1, \mathbb{C}^n)$.

3.1. DEFINITION. Let $F\ell^{(n)}$ be the set of $(L_i)_{i=-\infty}^{\infty}$ where L_i is a 1-dimensional subspace of $H^{(n)}$, L_i is perpendicular to L_j if $i \neq j$ and $\sum_{i=-\infty}^{\infty} L_i = H^{(n)}$.

Hence a map $\phi : M^2 \to F\ell^{(n)}$ can be described as $\phi = (\Pi_i)_{i=-\infty}^{\infty}$ with $\Pi_i : M^2 \to \mathbb{P}(H^{(n)})$ where $\mathbb{P}(H^{(n)})$ is the projective space formed by lines in $H^{(n)}$. Furthermore, $\Pi_i^2 = \Pi_i$, $\Pi_i \cdot \Pi_j = 0$ if $i \neq j$, $\Pi_i^* = \Pi_i$ and $\sum_i \Pi_i = I$.

From now on, we consider without mentioning the natural embedding of $F\ell^{(n)}$ into $\bigoplus_{i=-\infty}^{\infty} \mathbb{P}_i(H^{(n)})$, given by the fact that if $F \in F\ell^{(n)}$ then $F = (L_i)_{i=-\infty}^{\infty}$ where L_i are lines in $H^{(n)}$ so $F\ell^{(n)}$ embedds naturally into $\bigoplus_{i=-\infty}^{\infty} \mathbb{P}_i(H^{(n)})$.

We will now describe $F\ell^{(n)}$ in a algebraic fashion. We recall that each element of $H^{(n)} = L^2(S^1, \mathbb{C}^n)$ can be represented as $\sum_{k=-\infty}^{\infty} f_k Z^k$, $f_k \in \mathbb{C}^n$. $H^{(n)}$ always decomposes as $H^{(n)}_{(+)} \oplus H^{(n)}_{(-)}$ where $H^{(n)}_{(+)} = \{$functions whose negative Fourier coefficients vanish$\} = \{f \in H^{(n)}; f(Z) = \sum_{k \geqslant 0} f_k Z^k$ with $f_k \in \mathbb{C}^n\} = \{f \in H^{(n)}$; f is the boundary value of a function holomorphic in $|Z| < 1\}$ and $H^{(n)}_{(-)} = (H^{(n)}_{(+)})^{\perp} = \left\{f \in H^{(n)}; f(Z) = \sum_{k<0} f_k Z^k\right\}$.

Now to introduce an orthogonal basis in $H^{(n)}$; we basically identify $H^{(n)}$ with $H = L^2(S^1, \mathbb{C})$ with its natural basis $(e^{\sqrt{-1}k\theta})_{k \in \mathbb{Z}}$. Then, if we consider the natural transitive action of $L(U(n))$ on $F\ell^{(n)}$ and take the isotropy subgroup at the flag $\{\xi^k H_+\}$ which is a maximal torus T of $U(n)$, we obtain that $F\ell^{(n)} = L(U(n))/T$. See [24] for more details.

We also recall that any $\phi : S^1 \to L(u(n))$ can be represented by Fourier series as $\phi = \sum_{k=-\infty}^{\infty} a_k Z^k$, $a_{-k} = -a_k^*$ where $a_k \in u(n)_{\mathbb{C}}$. From now on, we will be using the fact that the coefficients a_k, for $k \geqslant 0$ determine ϕ completely.

Then, we can define the Killing form metric on $L(u(n))$ namely: if $\phi, \psi \in L(u(n))$ then $\langle \phi, \psi \rangle = \sum_{k \geqslant 0} tr(a_k b_k^*)$ where $\phi = \sum_{k=-\infty}^{\infty} a_k Z^k$ and $\psi = \sum_{k=-\infty}^{\infty} b_k Z^k$. We begin the discussion considering $F\ell^{(n)}$ equipped with the natural induced normal Killing form metric.

We define the energy of a map $\phi : M^2 \to F\ell^{(n)}$ as: $E(\phi) =$

$$\frac{1}{2} \sum_{i=0}^{\infty} \int_{M^2} \left(\left\langle \frac{\partial \Pi_i}{\partial x}, \frac{\partial \Pi_i}{\partial x} \right\rangle + \left\langle \frac{\partial \Pi_i}{\partial y}, \frac{\partial \Pi_i}{\partial y} \right\rangle \right) dx\,dy.$$

Let $q : M^2 \to L(u(n))$ and consider $\phi_t^i(x) = e^{-tq(x)} \cdot \Pi_i \cdot e^{tq(x)}$. Then

$$(\delta \Pi_i)(q)(x) = \frac{d}{dt}\Big|_{t=0} \phi_t^i(x) = [\Pi_i(x), q(x)].$$

3.2. PROPOSITION. *Let $\phi = (\Pi_i) : M^2 \to F\ell^{(n)}$ be a smooth map. Then ϕ is harmonic if and only if* $\sum_{i=0}^{\infty} [\Delta \Pi_i, \Pi_i] = 0$.

Proof. $(\delta E)(\delta\phi(q)) = \sum_i \int_{M^2} \left(\left\langle \frac{\partial \Pi_i}{\partial x}, \frac{\partial}{\partial x} (\delta \Pi_i(q)) \right\rangle + \left\langle \frac{\partial \Pi_i}{\partial y}, \frac{\partial}{\partial y} (\partial \Pi_i(q)) \right\rangle \right)$ $dxdy = \sum_{i=0}^{\infty} \langle\langle -\Delta\Pi_i, [\Pi_i, q]\rangle\rangle$. But according to the cyclic property of the trace, we see that $\langle\langle A, [B, C]\rangle\rangle = \langle\langle [B^*, A], C\rangle\rangle$. Therefore, $(\delta E)(\delta\phi(q)) = -\sum_{i=0}^{\infty} \langle\langle [\Pi_i, \Delta\Pi_i], q\rangle\rangle$.

But $\phi = (\Pi_i)_i : M^2 \to F\ell^{(n)}$ is harmonic if and only if it is a critical point of the energy functional; i.e. for any variation $\delta\phi(q)$ of ϕ we must have $(\delta E)(\delta\phi(q)) \doteq 0$. Hence if we apply Nöether's theorem and the fundamental lemma of the calculus of variations we see that $\phi : M^2 \to F\ell^{(n)}$ is harmonic if and only if $\sum_{i=0}^{\infty} [\Delta\Pi_i, \Pi_i] = 0$. ∎

Now let T_i be the tautological line bundle over $\mathbb{P}_i(H)$. Then, we consider the trivial vector bundle $M^2 \times H^{(n)}$ over M^2, so each $\phi^*(T_i)$ is a rank one subbundle of $M^2 \times H^{(n)}$. Let $\frac{\partial \Pi_i}{\partial x} = \partial \Pi_{i(\partial/\partial x)}$ be the covariant derivative of Π_i with respect to x and A_x^i the projection of $\frac{\partial \Pi_i}{\partial x}$ onto $\Pi_i^\perp$. We call the partial second fundamental forms of $\phi = (\Pi_i)_i : M^2 \to F\ell^{(n)}$ the maps $A_x^{ij} = \Pi_i(A_x^j) = \Pi_i \frac{\partial \Pi_j}{\partial x}$. In a similar way, we define A_y^{ij}. See [2] or [21] for more details.

Now if we think of M^2 as a complex one-dimensional manifold, then we define $\frac{\partial \Pi_i}{\partial Z} = \frac{1}{2}\left(\frac{\partial \Pi_i}{\partial x} - \sqrt{-1}\, \frac{\partial \Pi_i}{\partial y} \right)$ and $\frac{\partial \Pi_i}{\partial Z} = \frac{1}{2}\left(\frac{\partial \Pi_i}{\partial x} + \sqrt{-1}\, \frac{\partial \Pi_i}{\partial y} \right)$. We also define $A_{\bar{Z}}^{ij} = \Pi_i \frac{\partial \Pi_j}{\partial \bar{Z}}$ and $A_Z^{ij} = \Pi_i \frac{\partial \Pi_j}{\partial \bar{Z}}$.

Now let $\mu = z$ or z and

$$A_\mu = \begin{bmatrix} 0 & A_\mu^{12} & A_\mu^{13} & \dots & A_\mu^{1n} & \dots \\ A_\mu^{21} & 0 & A_\mu^{23} & \dots & A_\mu^{2n} & \dots \\ \vdots & \vdots & \vdots & & \vdots & \\ A_\mu^{n1} & A_\mu^{n2} & A_\mu^{n3} & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & & \vdots & \end{bmatrix}$$

3.3. PROPOSITION $\frac{\partial}{\partial z}(A_{\bar z}) - \frac{\partial}{\partial \bar z}(A_z) = [A_z, A_{\bar z}] + [A_z, A_{\bar z}]_h$ *where* $[A_z, A_{\bar z}]_h$ ***denotes*** *the diagonal part of the matrix* $[A_z, A_{\bar z}]$.

Proof. See [21].

Combining 3.2 and 3.3. Propositions we can see that $\phi = (\Pi_i)_i : M^2 \to F\ell^{(n)}$

is harmonic if and only if $\sum_{ij} \frac{\partial}{\partial Z}(A_{\bar{Z}}^{ij}) + \frac{\partial}{\partial \bar{Z}}(A_Z^{ij}) = 0$. Therefore, we have just proved a conservation law formula, which states that if $\phi = (\Pi_i)_i : \mathbb{C}P^1 \to F\ell^{(n)}$ is harmonic then $0 = tr(A_\mu) = tr(A_\mu^2)$ when $\mu = Z$ or $\bar{Z}$.

On the other hand:

$$E(\phi) = \frac{1}{2} \sum_{i,j=1}^{n} \int_{M^2} |A_Z^{i,j}|_{V_g}^2 =$$

$$= \frac{1}{2} \sum_{(i,j)\in S^+} \int_{M^2} |A_Z^{i,j} Z|_{V_g}^2 + \frac{1}{2} \sum_{(i,j)\in S'} \int_{M^2} |A_Z^{i,j}|_{V_g}^2$$

where S^+ is a partition of $(\mathbb{N} \times \mathbb{N} - D)$ where $D = \{(k, k); k \in \mathbb{N}\}$ such that if $(i, j) \in S^+$ then $(j, i) \in S^+$, and S^- is the complement of S^+ in $(\mathbb{N} \times \mathbb{N} - D)$. We call S^+ a positive system in $\mathbb{N}$.

Furthermore, we see that a map $\phi = (\Pi_i)_i : M^2 \to F\ell^{(n)}$ is S^+-holomorphic if and only if there exists S^+ such that $A_Z^{ij} = 0 \ \forall (i, j) \in S^+$. ■

Let us give a brief account about a set of invariant metrics on $F\ell^{(n)} = L(U(n))/T$. We will denote by p the tangent space of $F\ell^{(n)}$ at (T), so $L(u(n)) = p \oplus h$.

3.4. PROPOSITION: *a) The set of invariant metrics of $F\ell^{(n)}$ is naturally isomorphic to the set of scalar products $\langle,\rangle$ which are invariant under the action of $Ad(T)$ on $L(U(n))$. b) A scalar product $\langle,\rangle$ is invariant under $Ad(T)$ if and only if for each $v \in h$, $ad(v)$ is skew-symmetric with respect to $\langle,\rangle$ if and only if $\langle[v, y], Z\rangle = -\langle y, [v, Z]\rangle \ \forall Z, y \in L(u(n))$ and $v \in h$.*

Proof: See [21] with the obvious modifications and use the fact that exp: $L(u(n)) \to L(U(n))$ is onto.

Now let $\phi = a_0 + \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} a_k Z^k \in F\ell_{(T)}^{(n)}$ where $a_0 \in F(n)_{(T)} = \left(\frac{U(n)}{T}\right)_{(T)}$, $a_k \in u(n)_{\mathbb{C}}$ and $a_{-k} = -a_k^* = \bar{a}_k$. Therefore the invariant inner products on $F\ell^{(n)}$ are a combination of the invariant inner products $g_{a_1,\dots,a_{n-1},a_1+a_2,\dots,a_1+\dots+a_{n-1}}$ (see [16] for more details) with the following inner products on p:

$$\langle \phi, \psi \rangle = \sum_{k>0} \alpha_k \, tr(a_k b_k^*), \ \alpha_k > 0$$

where $\phi = \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} a_k Z^k$ and $\psi = \sum_{\substack{k=-\infty \\ k \neq 0}}^{\infty} b_k Z^k$.

In fact, let us recall the class of left-invariant metrics on $F(n) = U(n)/T$. See [18] or [21] for more details. We have that $F(n)_{(T)} = q$ where $u(n) = q \oplus h$, h being the Lie algebra of T. Then if A, B are in q, we consider the inner product $\langle A, B\rangle_a = \sum_{ij} \text{trace}\,(a^{ij} E_i A E_j B^*)$ where

$$E_i = i\begin{pmatrix} 0 & \cdots & \cdots & \overset{i}{\cdots} & \cdots & 0 \\ & 0 & & & & \\ & & \ddots & & & \\ & & & 1 & & \\ & & & & 0 & \\ & & & & & \ddots \\ 0 & & & & & 0 \end{pmatrix} 0 \quad a = (a^{ij}),\ a^{ij} = a^{ji} > 0.$$

If we restrict out almost complex structures in $F(n)$ to the integrable one we can see that $g_{(a_1, \ldots, a_{n-1}, a_1 + a_2, \ldots, a_1 + \ldots + a_{n-1})}$ give all the left-invariant Kähler metrics on $F(n)$.

Now, if we consider compactible almost complex structures in $F(n)$ and $\Omega(U(n))$ and also consider Kähler metrics on $F(n)$ of the form $g_{(a_1, \ldots, a_{n-1}, a_1 + a_2, \ldots, a_1 + \ldots + a_{n-1})}$ and consider Kähler metrics on $\Omega(U(n))$ by making $\alpha_1 = \lambda > 0, \ldots, \alpha_n = n\lambda$ for $n > 0$. In this way we obtain several Kähler metrics on $F\ell^{(n)}$. The most natural Kähler metrics on $F\ell^{(n)}$ is given by taking $a_i = 1$ for $i = 1, \ldots, n-1$ and $\lambda = 1$. ■

In fact what we are doing is to put on $F(n)$ the induced metric from the natural holomorphic and totally geodesic embedding of $F(n)$ into $\Omega(U(n))$ which we will give more details in § 7.

§4. EXAMPLES OF MAPS INTO $F\ell^{(n)}$, $\Omega(U(n))$ AND A FORMULA FOR THE SECOND VARIATION OF THE ENERGY FOR A HARMONIC MAP $\phi : M^2 \to F\ell^{(n)}$

We now study some basic examples of harmonic maps $\phi : M^2 \to F\ell^{(n)}$ which were found basically by Eells and Wood [3] in the $F(n)$ case and relate these maps with the ones found by Uhlenbeck in [28]. To some extent perhaps these maps are the only examples of harmonic maps into periodic flag manifolds or Loop groups generated in an algebraic and geometric fashion.

Let us recall the construction of the Eells-Wood maps in the finite dimensional flag manifold case. See [18] or [21] for more details.

Let $h : M^2 \to \mathbb{C}P^{n-1}$ be a full (non-degenerate) holomorphic map. Then h is given locally by $u(Z) = [(u_0(Z), \ldots, u_{n-1}(Z))]$, so we define the $k-th$ associated curve of h called σ_k by: $\sigma_k : M^2 \to G_{k+1}(\mathbb{C}^n)$, where $\sigma_k(Z) =$ span $\{u(Z), u'(Z), \ldots, u^k(Z)\}$. We can see that σ_k is well-defined. Then we consider $h_k : M^2 \to \mathbb{C}P^{n-1}$ given by: $h_k(Z) = \sigma_k(Z)^{\perp} \cup \sigma_{k+1}(Z)$. We have the following important theorem due to Burns, Din, EElls, Glaser, Stora, Wood and Zakarewski.

4.1. THEOREM. *For each* $0 \leqslant k \leqslant n-1, h_k : M^2 \to \mathbb{C}P^{n-1}$ *is harmonic. Furthermore, given* $\phi : \mathbb{C}P^1 \to \mathbb{C}P^{n-1}$ *harmonic, then there exists a unique k,* $0 \leqslant k \leqslant n-1$ *and* $h : \mathbb{C}P^1 \to \mathbb{C}P^{n-1}$ *full and holomorphic such that* $\phi = h_k$.

Therefore, we have canonical maps $\psi : M^2 \to F(n) = U(n)/T$ called Eells-Wood maps given by: $\psi(x) = (h_0(x), \ldots, h_{n-1}(x))$. We can see that the Eells-Wood maps are holomorphic and harmonic with respect to any invariant metric defined on $F(n)$ in §3. Moreover $A^{ij}_{\bar{Z}} = 0$ unless i and j are consecutive integers between 1 and n.

If $h : M^2 \to \mathbb{P}(H^{(n)})$ is a map, then h is given locally by $u : U \subseteq M^2 \to H^{(n)}$ $p \to \sum_{k=-\infty}^{\infty} f_k(p)e^{\sqrt{-1}k\theta}$, where $f_k(p) \in \mathbb{C}^n$, $f_{-k}(p) = \overline{f_k(p)}$. If we consider h holomorphic and full we can imitate the Eells-Wood construction in our case, and it seem that 4.1. Theorem remains true in this case.

Now associated to each Eells-Wood map $\psi = (h_0, \ldots, h_{n-1}) : M^2 \to F(n)$, we have $\tilde{\psi} : M^2 \to F\Omega^{(n)}$ given by: $\tilde{\psi}(p) = \sum_{k=0}^{n-1} h_k(p)e^{\sqrt{-1}k\theta} + \sum_{k=n}^{\infty} e^{\sqrt{-1}k\theta}$.

Clearly $\tilde{\psi} = (h_0, \ldots, h_{n-1}, 1, 1, \ldots)$ is harmonic with respect to the Killing form metric on $F\Omega^{(n)}$ since ψ harmonic implies $[\Delta h_i, h_i] = 0$ $i = 0, \ldots, n-1$ furthermore $[\Delta 1, 1]$ is clarly zero. Furthermore, we can see that the maps ψ are harmonic with respect to all invariant metrics defined on §3. See [18] or [21] for more details. From now on we identify ψ and $\tilde{\psi}$ and call $\tilde{\psi}$ a Eells-Wood map. In §7 we will see that the projection of these maps on $\Omega(U(n))$ via the fibration $F(n) \ldots \to F\Omega^{(n)} \to \Omega(U(n))$ are the Uhlenbeck's maps found in [28].

Notice we know that the energy of a map $\phi = (\Pi_i)_{i=-\infty}^{\infty} : M^2 \to F\Omega^{(n)}$ with the respect to the metric $g_\alpha = (g_{\alpha_{ij}})$ is given by:

$$E(\phi) = \frac{1}{2} \int_{M^2} \sum_{i,j=0}^{\infty} \alpha_{ij} |A^{ij}_{\mu}|^2 V_g,$$

where $\alpha_{ij} = \alpha_{ji} > 0$ and $\mu = z$ or $\bar{z}$.

4.2. PROPOSITION. *Let* $\phi = (\Pi_i)_{i=-\infty}^{\infty} : M^2 \to (F\ell^{(n)}, g_{\alpha_{ij}})$ *be a harmonic map. Then*

$$I^{\phi}_{\alpha=(\alpha_{ij})}(q) = (\delta^2 E_\alpha)(\delta\Pi_i(q)) =$$

$$= 2\,Re\left\{ \int_{M^2} \left\langle \alpha^{ij}\,\Pi_i\, \frac{\partial q}{\partial \bar z}\, \Pi_j + [q.^{\alpha}\, a_z],\ \frac{\partial q}{\partial \bar z} \right\rangle V g \right\}$$

Proof: We have seen that

$$\delta E_\alpha = -2\,Re\left\{ \int_{M^2} \left\langle \alpha_{ij}\ A^{ij}_z,\ \frac{\partial q}{\partial \bar z} \right\rangle \right\}.$$

Hence:

$$\delta^2 E_\alpha = -2\,Re\left\{ \int_{M^2} \alpha_{ij} \left\langle [A^{ij}_z, q] - \Pi_i\, \frac{\partial q}{\partial \bar z}\, \Pi_j,\ \frac{\partial q}{\partial \bar z} \right\rangle V g \right\} =$$

$$-2\,Re\left\{ \int_{M^2} \left\langle [a^{\alpha}_z, q] - \alpha^{ij}\Pi_i\, \frac{\partial q}{\partial \bar z}\, \Pi_j,\ \frac{\partial q}{\partial \bar z} \right\rangle V_g \right\}$$

therefore

$$(\delta^2 E_\alpha)\,(\delta\phi(q)) = 2\,Re\left\{ \int_{M^2} \left\langle \alpha^{ij}\Pi_i\, \frac{\partial q}{\partial \bar z}\, \Pi_j + [q\,.\,a^{\alpha}_z],\ \frac{\partial q}{\partial \bar z} \right\rangle V_g \right\}. \ \blacksquare$$

§5. STABILITY OF HARMONIC MAPS INTO $F\ell^{(n)}$

We start this section by proving the following useful and key lemma in our subsequent discussion.

5.1. LEMMA. *Let* $\psi = (\Pi_i)_i : M^2 \to F\ell^{(n)}$ *be an Eells-Wood map. Consider* $\beta_{ij} = \alpha_{ij} + \epsilon_k$ *if i and j are consecutive integers between 1 and n and* $\beta_{ij} = \alpha_{ij}$ *otherwise. Then* $I^{\psi}_{\beta_{ij}}(q) = I^{\psi}_{\alpha_{ij}}(q) + 4\epsilon_1\,|\,\delta(A^{13}_z)(q)\,|^2 + \ldots + 4\epsilon_\ell\,|\,\delta(A^{in}_z)(q)\,|^2$ *where* $\epsilon_1, \ldots, \epsilon_\ell$ *are arbitrary real numbers such that* $\beta_{ij} = \alpha_{ij} + \epsilon_k$ *is bigger than zero. We can easily see that* $\delta(A^{ij}_\mu)(q) = [A^{ij}_\mu, q] - \Pi_i\, \frac{\partial q}{\partial \mu}\, \Pi_j$ *where* $\mu = z$ *or* $\bar z$

Proof. We have that if ψ is an Eells-Wood map then $A_z^{ij} = 0$ if i and j are not consecutive integers between 1 and n or i is bigger than n or j is bigger than n. Therefore:

$$I^{\psi}_{(\beta_{ij})}(q) = 2\,Re \left\{ \int_{M^2} \langle [q, \alpha_{12}(A_z^{12} + A_z^{21}) + \ldots \right.$$

$$+ \alpha_{n(n-1)}(A_z^{n(n-1)} + A_z^{(n-1)n}) + (\alpha_{13} + \xi_1)0 + 0 + \ldots]$$

$$+ \alpha_{12}\left(\Pi_1 \frac{\partial q}{\partial z} \Pi_2 + \Pi_2 \frac{\partial q}{\partial z} \Pi_1\right) + \ldots + \ldots$$

$$\left. + (\alpha_{ij} + \epsilon_\ell)\left(\Pi_i \frac{\partial q}{\partial z} \Pi_j\right) + \left(\Pi_j \frac{\partial q}{\partial z} \Pi_i\right) + \ldots, \frac{\partial q}{\partial z} \rangle V_g \right\} =$$

$$= I^{\psi}_{(\alpha_{ij})}(q) + 4\epsilon_1 \mid \delta(A_z^{13})(q) \mid^2 + \ldots + 4\epsilon_\ell \mid \delta(A_z^{1n}) \mid^2 .$$ ■

Now we can use the lemma above to analyze the effect on the index form if we perturb a Kähler metric on $F\ell^{(n)}$.

5.2. THEOREM. *Let $\alpha = (\alpha_{ij})$ be a Kähler metric on $F\ell^{(n)}$ and $\beta = (\beta_{ij})$ be the metric obtained from α_{ij} by making $\beta_{ij} = \alpha_{ij} + \epsilon_k$ if i and j are consecutive integers between 1 and n, and $\beta_{ij} = \alpha_{ij}$ otherwise, where $\epsilon_1, \ldots, \epsilon_\ell$ are non-negative and $\psi : M^2 \to F\ell^{(n)}$ is a Eells-Wood map. Then ψ is stable.*

Proof. We know that ψ is holomorphic with respect to the natural almost complex structure on $F\ell^{(n)}$, according to Lichnerowicz's remark we see that $I^{\psi}_{(\alpha_{ij})}$ is positive semidefinite. If we apply 5.1. Lemma we have:

$$I^{\psi}_{(\beta_{ij})}(q) = I^{\psi}_{\alpha_{ij}}(q) + 4\epsilon_1 \mid \delta(A_z^{13})(q) \mid^2 + \ldots + 4\epsilon_\ell \mid \delta(A_z^{1n})(q)\mid^2 \geqslant 0$$

since $\epsilon_1, \ldots, \epsilon_\ell$ are $\geqslant 0$. Therefore ψ is stable. ■

Now let $\phi = (\Pi_i)_i : M^2 \to F\ell^{(n)}$. We say that ϕ is full or non-degenerate if for each i, there exists j such that $A_\mu^{ij} \neq 0$ where $\mu = z$ or z. Then:

5.3. THEOREM. *Let $\alpha = (\alpha_{ij})$ be a Kähler metric on $F\ell^{(n)}$ and $\beta = (\beta_{ij})$ a perturbation of α given by: $\beta_{ij} = \alpha_{ij} - \epsilon_k$ if i and j are consecutive integers between 1 and n, and $\beta_{ij} = \alpha_{ij}$ otherwise, where $\epsilon_1, \ldots, \epsilon_\ell$ are > 0, $\beta_{ij} = \alpha_{ij} - \epsilon_k > 0$ and $\psi : M^2 \to F\ell^{(n)}$ is an Eells-Wood map. Then ψ is not stable.*

Proof. According to Lichnerowicz's remark we know that the index form of the energy functional has the same index form for the $\bar{\partial}$-energy when the metric is Kähler. But $g_{(\alpha_{ij})}$ is Kähler, then if q is a holomorphic variation in the same almost complex structure that ψ is holomorphic then $I^{\psi}_{(\alpha_{ij})}(q) = 0$. Hence if we apply 5.1. Lemma we have:

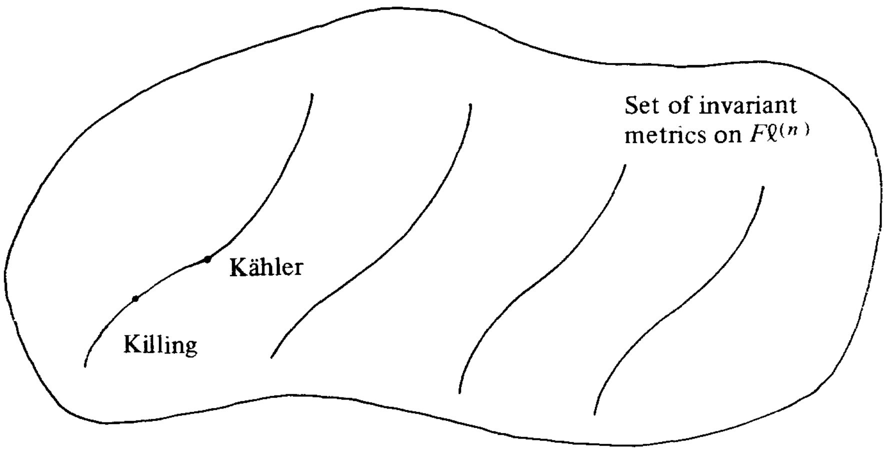
$$I^{\psi}_{(\beta_{ij})}(q) = I^{\psi}_{(\alpha_{ij})}(q) - 4\epsilon_1 \mid \delta(A_z^{13})(q) \mid^2 - \ldots - 4\epsilon_{\ell} \mid \delta(A_z^{1n})(q) \mid^2$$

Since ψ is full we choose a holomorphic variation q such that $\delta(A_z^{13})(q) \neq 0$ or ... $\delta(A_z^{1n})(q) \neq 0$, we have that: $I^{\psi}_{(\beta_{ij})}(q) < 0$. Hence ψ is not stable. We can see that such holomorphic variations always exist. ■

5.4. COROLLARY. *Let $\psi = (\Pi_i)_i : M^2 \to F\ell^{(n)}$ be a full Eells-Wood map, where $F\ell^{(n)}$ is equipped with the Killing form metric. Then ψ is not stable.*

Proof. Choose $\epsilon_1, \ldots, \epsilon_k$ such that $\beta_{ij} = \alpha_{ij} - \epsilon_k = 1$ for any integers i and j.■

Finally, we can visualize the set of invariant metrics on $F\ell^{(n)}$ in the following way:



The diagram shows that the Kähler metrics are saddles in the space of invariant metrics on $F\ell^{(n)}$.

§6. HARMONIC MAPS FROM $T^2 = S^1 \times S^1$ INTO $F\ell^{(n)}$

It is well known the existence of Yang-Mills connections over $S^2 \times S^2$ that are not instantons. It is also known a close relationship between the theory of equivariant Yang-Mills connections over Riemannian manifolds of cohomogeneity one and the theory of invariant minimal submanifolds of symmetric spaces. See [30] for more details.

In this paragraph we went to show the similarities of this theory with the study of harmonic two-tori into $F\ell^{(n)}$.

Consider the following circle action on $L(U(n))$ given by:

$$\rho : S^1 \times L(U(n)) \to L(U(n))$$

$$\left(e^{\sqrt{-1}\theta}, \sum_k a_k e^{\sqrt{-1}k}\right) \to \sum_k a_k e^{\sqrt{-1}}(k+\theta)$$

Assume further that the set of equivariant harmonic maps: $F_\rho = \{\phi \in C^\infty(S^1 \times \mathbb{R}, F\ell^{(n)});\ \phi(e^{\sqrt{1}\theta}, t) = \rho(e^{\sqrt{-1}\theta})\phi(t)$ where $\phi(t) = (\phi_i(t))_{i=-\infty}^{\infty}$ for all $e^{\sqrt{-1}\theta} \in S^1\}$ is nonempty. Note that $L(U(n))$ acts on $F\ell^{(n)}$ by conjugation:

$$L(U(n)) \times F\ell^{(n)} \to F\ell^{(n)}$$

$$(A, F) \to AFA^{-1}$$

Let $\phi = (\Pi_i)_i : S^1 \times \mathbb{R} \to F\ell^{(n)}$ given by $\phi(e^{\sqrt{-1}\theta}, t) = (\Pi_i(\theta, t))_i = (\exp(A\theta)\Pi_i(t))_i = \exp(A\theta)$. $\phi(t)$ where $\Pi_i's$ are projection operators and $A \in L(u(n))$. Note that $\Pi_i(\theta, t) = \exp(A\theta)\Pi_i(t)\exp(-A\theta)$.

Consider a local chart $U \subseteq \mathbb{R}^2$ for a Riemann surface M^2 and B_1, B_2 in $L(u(n))$ such that $[B_1, B_2] = 0$ that is $[B_1(e^{\sqrt{-1}\theta}), B_2(e^{\sqrt{-1}\theta})] = 0$ for any $e^{\sqrt{-1}\theta} \in S^1$. Then we can define locally the following map:

$$U \xrightarrow{\tilde{\phi}} L(U(n))$$

$$(x, y) \to \exp(B_1 x + B_2 y)$$

We see that $\tilde{\phi}$ induces a map $\phi = (\Pi_i)_i : U \to F\ell^{(n)}$ by: $\Pi_i = \tilde{\phi}^* = \exp(B_1 x + B_2 y) \cdot E_i \cdot \exp(-B_1 x - B_2 y)$ where $E_i = (a_{jk})$ where $a_{ii} = 1$ and $a_{jk} = 0$ if $j \neq k$.

Let us now compute the second fundamental forms of ϕ, namely: $^{ij}_x = a^{ij}A^{ij}_x$, $^{ij}_y = a^{ij}A^{ij}_y$, $a^{ij} > 0$. We have:

$$a^{ji}A^{ji}_x = a^{ji}\Pi_j \frac{\partial \Pi_i}{\partial x} = a^{ji}\Pi_j\{B_1 \exp(B_1 x + B_2 y)E_i \cdot \exp(-B_1 x - B_2 y)$$

$$-\exp(B_1 x + B_2 y)E_i B_1 \exp(-B_1 x - B_2 y)\}.$$

But since $B_1 \cdot B_2 = B_2 \cdot B_1$ we have

$$a^{ji}A^{ji}_x = a^{ji}_x = a^{ji}\{\exp(B_1 x + B_2 y)E_j \exp(-B_1 x - B_2 y) \cdot$$

$$\cdot (B_1 \exp(B_1 x + B_2 y)E_i \exp(-B_1 x - B_2 y) - \exp(B_1 x + B_2 y) \cdot$$

$$E_i B_i \exp(-B_1 x - B_2 y)\} = a^{ji}\exp(B_1 x + B_2 y)E_j B_1 E_i \cdot \exp(-B_1 x - B_2 y).$$

Now we can compute the Euler-Lagrange equations for such maps ϕ.

6.2. LEMMA. *Let* $\phi = (\Pi_i) : U \subseteq M^2 \to (F\ell^{(n)}, g_{a\,ij})$ *be a smooth map such that* $\Pi_i = \exp(B_1 x + B_2 y)E_i \exp(-B_1 x - B_2 y)$ *where* $B_1, B_2 \in L(u(n))$ *and* $[B_1, B_2] = 0$. *Then* ϕ *is harmonic if and only if* $\sum_{\substack{i,j\\ i\neq j}} a^{ij}E_i([B_1, \operatorname{diag} B_1] + [B_2, \operatorname{diag} B_2])E_j = 0$ *where diag* (B_i) *denotes the diagonal part of* B_i, $i = 1, 2$.

Proof. According to 3.2 Proposition ϕ is harmonic if and only if $\frac{\partial}{\partial x}(\mathscr{A}^a_x) + \frac{\partial}{\partial y}(\mathscr{A}^a_y) = 0$. Hence let us compute $\frac{\partial}{\partial x}(\mathscr{A}^a_x) + \frac{\partial}{\partial y}(\mathscr{A}^a_y)$. We have

$$\frac{\partial}{\partial x}(\mathscr{A}^a_x) = a^{ij}B_1 \exp(B_1 x + B_2 y)E_iB_1E_j \exp(-B_1 x - B_2 y) -$$

$$- a^{ij} \exp(B_1 x + B_2 y)E_iB_1E_jB_1 \exp(-B_1 x - B_2 y) =$$

$$= \exp(B_1 x + B_2 y)\ B_1, \sum_{\substack{i,j\\ i\neq j}} a^{ij}E_iB_1E_j \exp(-B_1 x - B_2 y) =$$

$$\exp(B_1 x + B_2 y)\, a^{ij}E_i [\operatorname{diag} B_1, B_1]\, E_j \exp(-B_1 x - B_2 y)$$

Similarly we prove that:

$$\frac{\partial}{\partial y}(\mathscr{A}^a_y) = \exp(B_1 x + B_2 y)a^{ij}E_i[\operatorname{diag} B_2, B_2]E_j\exp(-B_1 x - B_2 y)$$

Therefore $\frac{\partial}{\partial x}(\mathscr{A}^a_x) + \frac{\partial}{\partial y}(\mathscr{A}^a_y) = 0$ if and only if

$$\sum_{\substack{i,j\\ i\neq j}} a^{ij}E_i([B_1, \operatorname{diag} B_1] + [B_2, \operatorname{diag} B_2])\, E_j = 0. \qquad \blacksquare$$

6.3. THEOREM. *Let* $\phi = (\Pi_i)_i : \dfrac{\mathbb{R}^2}{\alpha\mathbb{Z} \oplus \beta\mathbb{Z}} = T^2 \to (F\ell^{(n)}, g_{a=(a\,ij)})$ *be an equivariant map like in 6.1. Lemma, where* $\Pi_i = \exp(B_1 x + B_2 y)E_i \exp(-B_1 x - B_2 y)$ *and* $B_1, B_2 \in L(u(n))$ *with* $[B_1, B_2] = 0$. *Furthermore, assume* $E_iB_kE_j \neq 0$ *for some* $1 \leqslant i \neq 1 \leqslant m$, $k = 1$ *or* 2 *and that* $\sum_{\substack{i,j\\ i\neq j}} a^{ij}E_i([B_1, \operatorname{diag} B_1] + [B_2, \operatorname{diag} B_2])E_j = 0$. *Then* ϕ *is harmonic with respect to the metric* $g_{a=(a\,ij)}$ *but is not holomorphic with respect to any almost complex structure on* $F\ell^{(n)}$.

Proof. According to our hypothesis and 6.2. Proposition we have that ϕ is harmonic. On the other hand, $A_z^{ij} = A_x^{ij} + \sqrt{-1}A_y^{ij}$ and $A_z^{ji} = A_x^{ji} + \sqrt{-1}A_y^{ji}$ are both non-zero as we have seen. So according to the holomorphic map equations in §3 ϕ is not holomorphic with respect to any almost complex structure on $F\ell^{(n)}$. ∎

Now let us study $f : \mathbb{R} \to L(U(n))$ where $f(t) = \exp(B^0 t)$ where $B^0 : S^1 \to U(n)$ is the constant function $B^0(\exp(\sqrt{-1}\theta)) = B$, for any $\exp(\sqrt{-1}\theta) \in S^1$, where

$$B = \begin{pmatrix} 0 & \alpha\sqrt{-1} & 0 & 0 & \dots & 0 & \dots \\ \alpha\sqrt{-1} & 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & 0 & \beta\sqrt{-1} & \dots & 0 & \dots \\ 0 & 0 & \beta\sqrt{-1} & 0 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \\ 0 & 0 & 0 & 0 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \end{pmatrix} \in u(n)\ n \geqslant 4$$

and α, β are non-zero real numbers.

Then

$$f(t) = \exp(B^0 t) = \begin{pmatrix} \cos\alpha t & 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & \cos\alpha t & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & \cos\beta t & 0 & \dots & 0 & \dots \\ 0 & 0 & 0 & \cos\beta t & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \\ 0 & 0 & 0 & 0 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \end{pmatrix}$$

$$+\sqrt{-1}\begin{pmatrix} 0 & \sin\alpha t & 0 & 0 & \dots & 0 & \dots \\ \sin\alpha t & 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & 0 & \sin\beta t & \dots & 0 & \dots \\ 0 & 0 & \sin\beta t & 0 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \\ 0 & 0 & 0 & 0 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \end{pmatrix}$$

Let us consider as the first set of examples the case $B_1 = B_2 = B$, where α and β are non-zero real numbers.

Now let us consider $\tilde{\phi} = \mathbb{R}^2 \to L(U(n))$

$$(x,y) \to \exp(Bx + By)$$

Then ϕ induces a map:

$$\phi : \frac{\mathbb{R}^2}{\frac{2\Pi}{\alpha}\mathbb{Z} \oplus \frac{2\Pi}{\beta}\mathbb{Z}} \to F\ell^{(n)} \text{ given by}$$

$$\left(x + \frac{2\Pi}{\alpha}n, y + \frac{2\Pi}{\beta}m\right) \overset{\phi}{\to} \tilde{\phi}(x, y)(E_1, \dots, E_n)\,\tilde{\phi}^*(x, y)$$

$$\exp(B(x+y))(E_1, \dots, E_m)\exp(-B(x+y))$$

But diag $B = 0$, so 6.3. Theorem says that ϕ is harmonic with respect to any invariant metric on $F\ell^{(n)}$ but is not holomorphic with respect to any almost complex structure on $F\ell^{(n)}$ since $E_1BE_2 = E_2BE_1 = \alpha\sqrt{-1} \neq 0$.

More generally we could take any B of the form

$$B = \begin{pmatrix} 0 & \alpha_1\sqrt{-1} & 0 & 0 & \dots & 0 & \dots & 0 & \dots & 0 & \dots \\ \alpha_1\sqrt{-1} & 0 & 0 & 0 & \dots & 0 & \dots & 0 & \dots & 0 & \dots \\ 0 & 0 & 0 & \alpha_2\sqrt{-1} & \dots & 0 & \dots & 0 & \dots & 0 & \dots \\ 0 & 0 & \alpha_2\sqrt{-1} & 0 & \dots & 0 & \dots & 0 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & & & & & & & \\ 0 & 0 & 0 & 0 & \dots & 0 & \dots & \alpha_k\sqrt{-1} & 0 & \dots & \\ 0 & 0 & 0 & 0 & \dots & \alpha_k\sqrt{-1} & \dots & 0 & 0 & \dots & \\ \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \\ 0 & 0 & 0 & 0 & & 0 & & 0 & 0 & \dots & \\ \vdots & \vdots & \vdots & \vdots & & \vdots & & \vdots & \vdots & & \end{pmatrix}$$

such that $2k \leqslant n$.

Another family of harmonic maps with respect to any invariant metric on $F\mathfrak{L}^{(n)}$ is given by taking:

$$B_1 = \begin{pmatrix} 0 & \alpha\sqrt{-1} & 0 & 0 & \dots & 0 & \dots \\ \alpha\sqrt{-1} & 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & 0 & \beta\sqrt{-1} & \dots & 0 & \dots \\ 0 & 0 & \beta\sqrt{-1} & 0 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \\ 0 & 0 & 0 & 0 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \end{pmatrix}$$

and

$$B_2 = \begin{pmatrix} 0 & \beta\sqrt{-1} & 0 & 0 & \dots & 0 & \dots \\ \beta\sqrt{-1} & 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & 0 & 0 & \alpha\sqrt{-1} & \dots & 0 & \dots \\ 0 & 0 & \alpha\sqrt{-1} & 0 & \dots & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \end{pmatrix}$$

where α, β are non-zero real numbers such that $\alpha/\beta \in Q$. Then $B_1, B_2 \in L(u(n))$, $[B_1, B_2] = 0$ and furthermore there exists $\gamma \in \mathbb{R}$ such that $\alpha \cdot \gamma$ and $\beta \cdot \gamma$ are integers.

Now let us consider $\phi : \dfrac{\mathbb{R}^2}{2\Pi\gamma(\mathbb{Z} \oplus \mathbb{Z})} \to F\mathfrak{L}^{(n)}$ given by $\phi(x + 2\Pi\gamma n, y + 2\Pi\gamma m) = \tilde{\phi}(x, y)(E_1, \dots, E_n)\tilde{\phi}^*(x, y) = \exp(B_1 x + B_2 y)(E_1, \dots, E_n)\exp(-B_1 x - B_2 y)$. But diag $(B_1) =$ diag $(B_2) = 0$. Then again using 4.3. Theorem we see that ϕ is harmonic with respect to all invariant metrics defined on § 2 but not holomorphic since $E_1 B_1 E_2 = E_2 B_1 E_1 = \alpha\sqrt{-1} \neq 0$.

We can generalize this example by taking $B_1 \in u(n)$ of the following form:

$$B_1 = \begin{pmatrix} B_i^1 & 0 & \dots & 0 & \dots & 0 \\ 0 & B_2^i & & \cdot & & \cdot \\ \vdots & \vdots & \ddots & \vdots & & \vdots \\ 0 & 0 & \dots & B_i^k & & \\ \vdots & \vdots & & & \ddots & \\ 0 & 0 & \dots & 0 & \dots & 0 \end{pmatrix}$$

where

$$B_1^i = \begin{pmatrix} 0 & \alpha_i\sqrt{-1} & 0 & 0 \\ \alpha_i\sqrt{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta_i\sqrt{-1} \\ 0 & 0 & \beta_i\sqrt{-1} & 0 \end{pmatrix}$$

$$B_2 \quad \begin{pmatrix} B_2^1 & 0 & 0 & 0 & 0 \\ 0 & \ddots & 0 & 0 & 0 \\ 0 & 0 & B_2^k & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

where

$$B_2^i = \begin{pmatrix} 0 & \beta_i\sqrt{-1} & 0 & 0 \\ \beta_i\sqrt{-1} & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_i\sqrt{-1} \\ 0 & 0 & \alpha_i\sqrt{-1} & 0 \end{pmatrix}$$

such that $2k \leqslant n$. Furthermore, we also assume $\alpha_1/\beta_1 = \dots = \alpha_k/\beta_k$ are all rational numbers.

§ 7. HARMONIC MAPS INTO LOOP GROUPS

The articles of Atiyah [3] and Donaldson [12] suggest a possible connection between harmonic 2-spheres into Loop groups and Yang-Mills connections over

S^4. The articles [7, 15, 28 and 30] suggest a possible way of approaching these questions which we will follow in this paragraph.

Let us recall the maps found by Uhlenbeck in [28]. If $\gamma : S^1 \to U(n)$ is a closed geodesic, i.e. $\gamma(t) = \exp(t\xi)$ where $\xi \in u(n)$ and $\exp(2\pi\xi) = I$.

We can consider the following action of $U(n)$ onto $\Gamma = Hom(S^1, U(n))$

$$U(n) \times \Gamma \to \Gamma$$

$$(g, \gamma) \to g\gamma g^{-1}$$

If we fix $\gamma \in \Gamma$, then a $U(n)$ - orbit of γ is of the following form: $U(n) \cdot \gamma = Ad(G)\xi/H$ where $H = \ g \in U(n); g\xi g^{-1} = \xi\ $.

Now we define the following embedding of $U(n)/H$ into $\Omega(U(n))$ namely:

$$\frac{U(n)}{H} \overset{\kappa}{\to} \Omega(U(n))$$

$$gH \overset{\kappa}{\to} g\gamma g^{-1}$$

If we put on $U(n)/H$ the metric and the almost complex structure induced by this natural embedding into $\Omega(U(n))$ we see that is holomorphic and totally geodesic. See [22] for more details and a proof of this fact.

Not let us consider

$$\xi = \begin{pmatrix} \lambda_1\sqrt{-1} & 0 & & 0 \\ 0 & \lambda_2\sqrt{-1} & & 0 \\ 0 & 0 & \ddots & 0 \\ 0 & 0 & & \lambda_n\sqrt{-1} \end{pmatrix}$$

$\lambda_1 \neq \lambda_2 \neq \ldots \neq \lambda_n, \lambda_i \in \mathbb{Z}$ so $\exp(2\Pi\xi) = I$. Then:

$$\exp(t\xi) = \exp(\lambda_1 \pm\sqrt{-1})\, E_1 + \ldots + \exp(\lambda_n \pm\sqrt{-1})\, E_n =$$
$$Z^{\lambda_1} E_1 + \ldots + \ldots + Z^{\lambda_n} E_n$$

Now let

$$\Pi_i : M^2 \to \Omega(U(n))$$

$$p \to \sum_i Z^{\lambda_i}\, \Pi_i(g)(p) = \sum_i Z^{\lambda_i} g(p)\, E_i g(p)^*$$

where $g : M^2 \to U(n)$ and $\psi = (\Pi_1, \ldots, \Pi_n) : M^2 \to F(n) = U(n)/T$ are the Eells-Wood maps.

Clearly each Eells-Wood map $\phi = (\Pi_1, \ldots, \Pi_n) : M^2 \to F(n)$ determines a map $\tilde{\phi} = K \circ \phi : M^2 \to \Omega(U(n))$ where $\tilde{\phi}(p) = \sum_{i=1}^{n} Z^{\lambda_i} g(p) E_i g(p)^*$.

Now we recall the following basic fact:

7.1. PROPOSITIONS. *Suppose that*

$$\begin{array}{ccc} M & \xrightarrow{\psi} & N \\ \rho \downarrow & & \downarrow \Pi \\ P & \xrightarrow{\phi} & Q \end{array}$$

is a commutative diagram, where ρ *and* Π *are Riemannian submersions with* $d\psi(T(M)^H) \subseteq T(N)^H$ *where* $T(M)^H_x = \ker(d\rho(x))^\perp$. *Assume that one of the following conditions is satisfied*

(a) $d\psi(T(M)) \subseteq T(N)^H$

(b) Π *has totally geodesic fibres*

(c) for all P, $\rho^{-1}(z) \to \Pi^{-1}(\phi(z))$ *is a Riemannian fibration with minimal fibres. Then* $\Pi \circ \psi$ *is harmonic if and only if* $\tau(\psi)$ *is vertical (where* $\tau(\psi)$ *is called the tension field of* ψ*, and* $\boldsymbol{\tau}(\psi) = 0$ *if and only if* ψ *is harmonic).*

Proof. See [14].

We have studied in §5 and §6 some special examples of harmonic maps from M^2 to $F\ell^{(n)}$. Now we connect our study of harmonic maps into periodic flag manifolds with harmonic maps into Loop groups by using the Grassmannian model of Loop groups as in [24].

Now considering the fibration $F(n) \ldots \to F\ell^{(n)} \to \Omega(U(n))$ we know that the Eells-Wood maps $\psi : M^2 \to F\ell^{(n)}$ are horizontal with respect to Π. that is $d\psi(T(M)) \subset T(F\ell^{(n)})^H$. Furthermore for each ψ, $\tau(\psi) = 0$ since ψ is harmonic. Hence $\Pi \circ \psi : M^2 \to \Omega(U(n))$ define new harmonic maps which we call Uhlenbeck's maps.

We imagine that by using Uhlenbeck's maps we can see that the set of invariant metrics on $\Omega(U(n))$ has also the same saddle phenomenon that we have proved in §6 for a precise set of invariant metrics on $F\ell^{(n)}$.

In fact by using O'Neill's techniques and the formula for the second variation of the energy we can see that part of the above conjecture is true.

Let M and N smooth manifolds and $\Pi : M \to N$ be a smooth map such that $\Pi^{-1}(p)$ is a smooth k-dimensional submanifold of M for all $p \in N$. Let $V = \Pi^{-1}(p)_q$ i.e. the tangent space to $\Pi^{-1}(p)$ at $q \in \Pi^{-1}(p)$. Assume that M and N have Riemannian metrics and set $H = V^\perp$. H and V are called the horizontal and vertical subspaces, respectively, and we use H and V as superscrits to denote horizontal and vertical components. Π is called a Riemannian submersion if $d\Pi \mid H$ is an isometry. If $X \in \chi(N)$, then there exists a unique $\overline{X} \in \chi(M)$ such that $d\Pi(X) = \overline{X}$ and $\overline{X} \in H$.

If K_M, K_N denote the sectional curvatures of M and N, respectively then we have the following well-known result due to O'Neill.

7.2. THEOREM. $K_N(\overline{X}, \overline{Y}) = K_M(\overline{X}, \overline{Y}) + \frac{3}{4} \| [\overline{X}, \overline{Y}]^V \|^2$

Proof. See [15]. ∎

On the other hand, if $\phi : M \to N$ is harmonic then:

$$I^{\phi}_{(q)} = \int_M \left\{ \left\langle \frac{\partial q}{\partial x}, \frac{\partial q}{\partial x} \right\rangle + \left\langle \frac{\partial q}{\partial y}, \frac{\partial q}{\partial y} \right\rangle - K_N\left(\frac{\partial \phi}{\partial x}, q \right) - K_N\left(\frac{\partial \phi}{\partial y}, q \right) \right\} V_g$$

Now we use the expression above in the case that $\psi : M \to N$ is harmonic map. $\Pi : N \to P$ is Riemannian submersion, $\Pi \circ \psi : M \to P$ is harmonic and ψ is horizontal with respect to Π. Then we have:

$$I^{\Pi \circ \psi}(q) = \int_M \left\{ \left| \frac{\partial \bar{q}}{\partial x} \right|^2 + \left| \frac{\partial \bar{q}}{\partial y} \right|^2 - K_P\left(d\Pi \circ d\psi \frac{\partial}{\partial x}, \bar{q} \right) - \right.$$

$$\left. - K_P\left(\frac{\partial^2 (\Pi \circ \psi)}{\partial y^2}, q \right) \right\} V_g \leqslant$$

$$\leqslant \int_M \left\{ \left| \frac{\partial \bar{q}}{\partial x} \right|^2 + \left| \frac{\partial \bar{q}}{\partial y} \right|^2 - K_N\left(\left(\frac{\partial \psi}{\partial x} \right), \bar{q} \right) - K_N\left(\left(\frac{\partial \psi}{\partial y} \right), q \right) \right\} V_g = I^{\psi}(q)$$

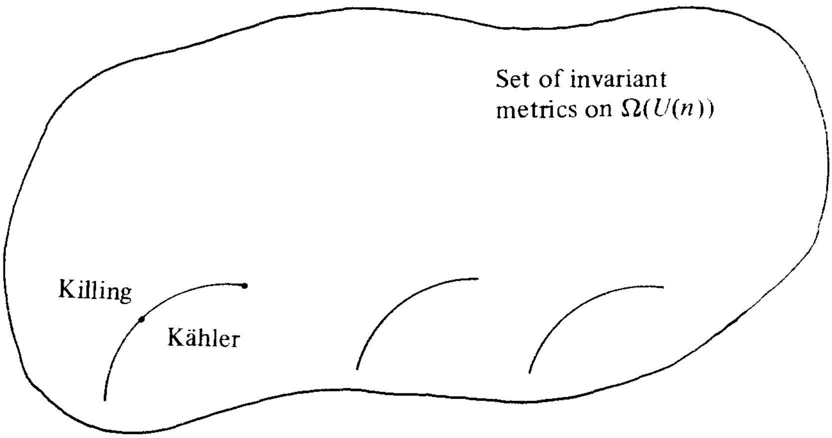
where q is a horizontal variation.

If we specialize to the case that we are interested in this note we have: $M = M^2$ is any oriented compact Riemann surface, $N = F\ell^{(n)} \cong \frac{L(U(n))}{T}$, $P = \Omega(U(n)) \cong \frac{L(U(n))}{U(n)}$ and $\psi : M^2 \to F\ell^{(n)}$ is a Eells-Wood map. We have $\Pi : F\ell^{(n)} \to \Omega(U(n))$ is a Riemannian submersion and ψ is horizontal with respect to Π. So if q is a horizontal variation, $I^{\Pi \circ \psi}(q) \leqslant I^{\psi}(q)$. Summarizing the results above we have:

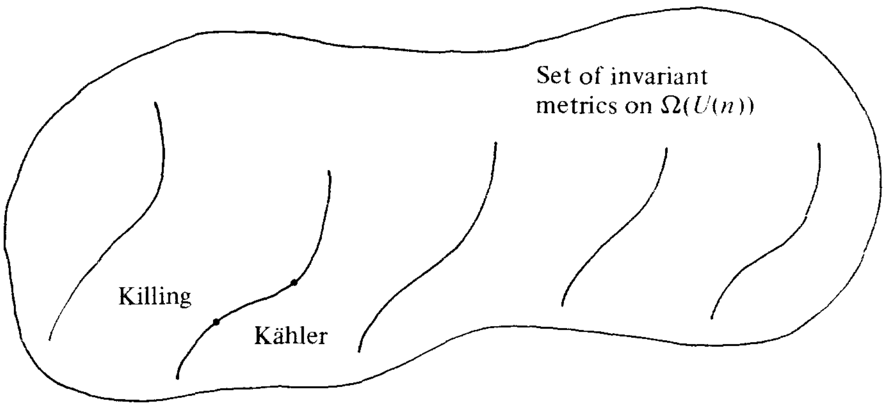
7.3. THEOREM. *Let* $a = (a^{ij})$ *be a metric on* $F\ell^{(n)}$ *such that the harmonic map* $\psi : M^2 \to (F\ell^{(n)}, g_a)$ *is not stable. Then the Uhlenbeck map* $\Pi \circ \psi : M^2 \to$

$(\Omega(U(n)), g_{a=(a^{ij})}$ *is also not stable.*

Proof. Take q any horizontal variation such that $I^{\psi}_{(a^{ij})}(q) < 0$. Then $I^{\Pi\circ\psi}_{(a^{ij})}(q) \leqslant I^{\psi}_{(a^{ij})}(q) < 0$, therefore $\Pi \circ \psi$ is not stable. We can picture the whole situation as:



We conjecture that the actual situation is described by the following picture.



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